Math 53, Discussions 116 and 118 2022-8-29

Parametric curves and their tangents II

Answers included

Conceptual questions

Question 1. We've seen that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

provided that the denominator is nonzero. How do we compute d^2y/dx^2 for a parametric curve?

Question 2. Let P be a point on a curve where the tangent is neither horizontal nor vertical. True or false:

- If $dy/dx > 0$ at P, then $dx/dy > 0$ also.
- If $d^2y/dx^2 > 0$ at P, then $d^2x/dy^2 > 0$ also.

Question 3. Given any single-variable function $f(x)$, you can view its graph as a curve in the xy-plane, with Cartesian equation $y = f(x)$. How can you parametrize this curve? Suppose that (x_0, y_0) is a point on the graph. If you compute the slope at this point using the §10.2 formula, do you get the same answer as you expect from Math 1A?

Computations

Problem 1. Find the equation of the tangent line to the parametric curve

$$x = 4e^{t-3} + 1 \qquad y = \sin(\pi t) + t \qquad -\infty < t < \infty$$

at the point $(5, 3)$.

Problem 2. You're probably sick of parametrizing circles at this point, but here's one parametrization of $x^2 + y^2 = 1$ that might be new to you, called *stereographic projection*.

The line passing through the points $(0,1)$ and $(t,0)$ intersects the unit circle at one point other than $(0,1)$. Find the coordinates of this point, in terms of t.

In the parameter interval $-\infty < t < \infty$, does this parametrization trace out the entire circle?

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1. Just as how the chain rule yields

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$$

we likewise have

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x}}{\mathrm{d}x/\mathrm{d}t}.$$

I did an explicit example of this calculation in discussion.

Question 2.

- This is true. The Inverse Function Theorem says that these two quantities are reciprocals of one another.
- This is false. Consider the curve $y = x^2$ at the point $(1,1)$. You can compute that $\mathrm{d}^2y/\mathrm{d}x^2 = 2 > 0$ at this point, whereas $\mathrm{d}^2x/\mathrm{d}y^2 = -1/4 < 0$. Geometrically, this means that the curve $y = x^2$ is concave up, but when reflected across the line $y = x$ (i.e. if you interchange the x, y-axes) it becomes concave down.

Question 3. All I really meant by this question was to remind you of the fact that any graph $y = f(x)$ can be simply parametrized as $x = t, y = f(t)$, and that the formula for slope for parametric curves in this case yields $f'(x_0)/1 = f'(x_0)$, as you expect from single-variable calculus.

Answers to computations

Problem 1. This is a straightforward computation so I will just provide the answer:

$$y - 3 = \frac{1-\pi}{4}(x-5).$$

Problem 2. The equation of the line through the points $(0,1)$ and $(t,0)$ is given by

$$x = t - ty.$$

To find where this intersects the unit circle, we plug this into the circle equation to obtain

$$(t-ty)^2 + y^2 = 1$$

which expands to the following quadratic:

$$(t^2+1)y^2 - 2t^2y + (t^2-1) = 0.$$

We already know one solution to this quadratic because the line clearly intersects the circle at $(0,1)$. So $y = 1$ is a solution. By factoring out $y - 1$, you will discover that the other solution is

$$y = \frac{t^2-1}{t^2+1}$$

which means

$$x = \frac{2t}{t^2+1}.$$

This is our desired parametrization. In the parameter interval $-\infty < t < \infty$, this traces out the circle once, but it misses the point $(0,1)$. As $t \to -\infty$, we approach $(0,1)$ from the left. As $t \to \infty$, we approach $(0,1)$ from the right.